University of Groningen
Exam Numerical Mathematics 1, July 5, 2016

Use of a simple calculator is allowed. All answers need to be motivated.
In front of the exercises you find its weight. In fact it gives the number of tenths which can be gained in the final mark. In total 5.4 points can be scored with this exam.

Exercise 1

(a) The error in polynomial interpolation of a function $f(x)$ on the points x_k, $k = 0, \cdots n$ is of the form $q(x)g(x)$ where $q(x)$ is a polynomial and, in general, $g(x)$ is a function.

- (i) [2] Show that $q(x_k)g(x_k) = 0$ for $k = 0, \cdots n$.
- (ii) [2] In general $g(x_k) \neq 0$ for $k = 0, \cdots n$. Use this to derive the polynomial $q(x)$.
- (iii) [1] Give also the factor $g(x)$ of the error.

(b) On the interval $[0, 3]$ a function $f(x)$ is interpolated at the points 0 and 2.

- (i) [4] Derive the interpolation polynomial.
- (ii) [3] Derive the integration rule for an integral over the interval [0,3] based on the interpolation polynomial of the previous part.
- (iii) [2] What is the degree of exactness of the thus constructed rule?

Exercise 2

Say A is a matrix of order 10 and it has the spectrum $\sigma(A) = \{1, 2, 3, 4, ..., 10\}$. Consider the iteration $x^{(0)}$ given, $y^{(n+1)}$ follows from solving $(A - 2.1I)y^{(n+1)} = x^{(n)}$ and $x^{(n+1)} = y^{(n+1)}/||y^{(n+1)}||$, for $n = 0, 1, 2, ...$

(a) [5] Where will $x^{(n)}$ converge to for $n \to \infty$?

(b) [1] What is the speed of convergence eventually?

(c) [4] Suppose instead of 2.1 we have the value 2.5. Show that $x^{(2n)}$ and $x^{(2n+1)}$ converge both, but each to a different vector.

(d) [2] Suppose moreover that A is symmetric. Explain that one cannot use the Cholesky factorization to solve the system for $y^{(n+1)}$? Also define the Cholesky factorization.

(e) [3] LU factorization with partial pivoting can be applied to solve the system. Explain what this factorization is and in particular what partial pivoting is.

Exercise 3

Consider the fixed point iteration $x^{(n+1)} = \phi(x^{(n)})$ with $x^{(0)}$ given and $\phi(x)$ a one times differentiable function.

(a) [2] Show that for convergence we must have that $|\frac{d\phi(x)}{dx}| < 1$ in the neighborhood of the fixed point.

(b) Let $\phi(x) = x + a(x)f(x)$.

- (i) [1] For which $a(x)$ is the fixed point of the above iteration the zero of $f(x)$?
- (ii) [2] What is the best choice we can make for $a(x)$ in order to get the fastest convergence?

(c) [3] Suppose that the fixed point method defined above is generalized to vectors, so $x \in R^n$ and $\phi(x)$ is a map from R^n into itself. Give the generalization of the criterion of part (a) to systems and derive this.

Exercise 4

Consider the initial value problem

$$\frac{dy}{dt} = f(y,t), \quad y(t_0) = y_0$$

Consider the trapezoidal method $w^{(n+1)} = w^{(n)} + \frac{\Delta t}{2}[f(w^{(n)}, t_n) + f(w^{(n+1)}, t_{n+1})]$, with $t_n = t_0 + n\Delta t$.

(a) $\boxed{5}$ Show that truncation error of this method is $\tau_n(\Delta t) = O(\Delta t^2)$.

(b) $\boxed{4}$ Show that the region of absolute stability of this method is the left half of the complex plane.

Consider on [0,1] and for $t > 0$ the parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin(\pi x t) \tag{1}$$

with $u(x,0) = sin(\phi x), u(0,t) = u(1,t) = 0$.

(c) $\boxed{5}$ Show that the space discretization on a grid with mesh size $h = 1/(m+1)$ leads to a coupled system of ordinary differential equations given by

$$\frac{dv}{dt} = Av + F(t),$$

where $A = \frac{1}{h^2}(B - 2I)$ with

$$B = \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}.$$

Give also $F(t)$

(d) $\boxed{2}$ For general A it holds that $\rho(A) \leq ||A||_\infty$ where $||A||_\infty = \max_i \sum_{j=1}^{m} |a_{ij}|$. Show that for the specific A defined in the previous item the eigenvalues are in the interval $[-\frac{4}{h^2}, 0]$.

(e) $\boxed{1}$ Show that the trapezoidal method applied to the system of ordinary differential equations defined in part (c) does not have a restriction on the time step as far as stability is concerned.